Important Formulas From Math 211

Note: This is not guranteed to be a comprehensive list.

Gaussian Elimination

Three Operations:

- Add a multiple of a row to another row (e.g. $R_2 - 2R_1 \to R_2$)
- Multiply a row by a nonzero number (e.g. $3R_2$)
- Interchange two rows (e.g. $R_2 \leftrightarrow R_3$)

Gaussian Elimination

$$p = n - r$$

p: Number of parameters in the solution
n: Number of variables in the system (number of columns in the coefficient matrix)
r: Rank (number of leading ones in **row echelon form**)

Gaussian Elimination

Row echelon form:

- All zero rows at the bottom
- Every non-zero row starts with a leading one
- All entries below and to the left of a leading one are zero

$$\begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Gaussian Elimination

Reduced row echelon form:

- All zero rows at the bottom
- Every non-zero row starts with a leading one
- All entries below and to the left of a leading one are zero
- **All entries above a leading one are zero**

$$\begin{bmatrix} 1 & * & 0 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Gaussian Elimination

Method:

1. Get a leading one in the top-left position
2. Use the leading one to get zeros beneath it
3. Move to the next row and column and repeat
4. Use the leading ones to get zeros above them

The Inverse of a Matrix

- $A^{-1}A = I = AA^{-1}$
- Finding the inverse:

$$[\ A \mid I\ ] \to \left[I \mid A^{-1}\ \right]$$

OR

$$A^{-1} = \frac{1}{\det A}\,\mathrm{adj}\, A$$

Properties

- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $(kA)^T = kA^T$

- $(A+B)^{-1} = ???$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(kA)^{-1} = \frac{1}{k}A^{-1}$

- $(A^T)^{-1} = (A^{-1})^T$

Determinants

- $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$
- Cofactor expansion along a row or a column (Remember the sign $(-1)^{i+j}$)
- Row Operations:
  - $R_m + kR_n$: No change to determinant
  - kR_m: Multiplies determinant by k
  - $R_m \leftrightarrow R_n$: Changes sign of determinant

Properties of Determinants

- $\det(A^T) = \det A$
- $\det(AB) = \det A \det B$
- $\det(A^{-1}) = \dfrac{1}{\det A}$
- $\det(kA) = k^n \det A$ if A is $n \times n$

Eigenvalues and Eigenvectors

- $X(\neq 0)$ is an eigenvector with corresponding eigenvalue λ if and only if
$$AX = \lambda X$$
- Characteristic Polynomial: $c_A(x) = \det(xI - A)$
- Finding Eigenvalues: Solve the equation $\det(\lambda I - A) = 0$
- Finding Eigenvectors for λ: Solve the system $(\lambda I - A)X = 0$

Diagonalization

$$P^{-1}AP = D$$

- $P = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}$
- $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$

Linear Transformations

- $T(X+Y) = T(X) + T(Y)$
- $T(kX) = kT(X)$

If A is the standard matrix for S, and B is the standard matrix for T, then:

- AB is the standard matrix for $S \circ T$ (T followed by S)
- A^{-1} is the standard matrix for S^{-1}

The standard matrix for a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is

$$\begin{bmatrix} T\begin{bmatrix}1\\0\end{bmatrix} & T\begin{bmatrix}0\\1\end{bmatrix} \end{bmatrix}$$

Linear Transformations

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

- Rotation (counterclockwise) through an angle of θ:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- Projection onto the line $y = mx$:

$$P_m = \frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$$

- Reflection across the line $y = mx$:

$$Q_m = \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

Complex Numbers

- $i^2 = -1$
- $\overline{a + bi} = a - bi$
- Dividing complex numbers: multiply numerator and denominator by the conjugate of the denominator.
- $|a + bi| = \sqrt{a^2 + b^2}$

Polar Form

$$a + bi \quad \leftrightarrow \quad re^{i\theta}$$

- $r = \sqrt{a^2 + b^2}$
- $\cos\theta = \dfrac{a}{r}$
- $\sin\theta = \dfrac{b}{r}$
- $e^{i\theta} = \cos\theta + i\sin\theta$

Vectors

- $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ (coordinates of B - coordinates of A)
- $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$
- $\overrightarrow{AB} = -\overrightarrow{BA}$

Dot Product

$$X \cdot Y = X^T Y$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax + by + cz$$

- $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta$
- $\vec{u} \cdot \vec{v} = 0$ if and only if $\vec{u} \perp \vec{v}$
- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

$$\operatorname{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$$

$$\| \operatorname{proj}_{\vec{v}} \vec{u} \| = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}$$

Cross Product

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det \begin{bmatrix} \hat{i} & a & x \\ \hat{j} & b & y \\ \hat{k} & c & z \end{bmatrix}$$

- $\vec{u} \times \vec{v} \perp \vec{u}$
- $\vec{u} \times \vec{v} \perp \vec{v}$
- $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta$

Planes

- Normal vector $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$
- Point on the plane $P_0(x_0, y_0, z_0)$
- Arbitrary Point $P(x, y, z)$
- Equations:

$$
\begin{aligned}
&ax + by + cz = d \\
&a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \\
&\vec{n} \cdot (P - P_0) = 0
\end{aligned}
$$

Lines

- Direction vector $\vec{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$
- Point on the line $P_0(x_0, y_0, z_0)$
- Arbitrary Point $P(x, y, z)$
- Equations:
  - Vector form:

$$\begin{array}{ccccc} P & = & P_0 & + & t & \vec{d} \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} & = & \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} & + & t & \begin{bmatrix} a \\ b \\ c \end{bmatrix} \end{array}$$

  - Parametric form:

$$\begin{array}{rcl} x & = & x_0 + ta \\ y & = & y_0 + tb \\ z & = & z_0 + tc \end{array}$$