MATH 211 review notes

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1 Determinants

1. Let $A = \begin{pmatrix} 2 & 1 & 2 \\ -3 & -1 & -1 \\ 5 & 2 & 1 \end{pmatrix}$.

   (a) Compute $\det(A)$

   Solution: To find the determinant, we expand along the first row.

$$\begin{aligned} \det(A) &= 2\begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} -3 & -1 \\ 5 & 1 \end{vmatrix} + 2\begin{vmatrix} -3 & -1 \\ 5 & 2 \end{vmatrix} \\ &= 2(-1+2) - 1(-3+5) + 2(-6+5) \\ &= 2 - 2 - 2 \\ &= -2 \end{aligned}$$

   (b) Compute $adj(A)$

   Solution: First we find the cofactor matrix of A.

$$\begin{aligned} cof(A) &= \begin{pmatrix} \begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix}, & -\begin{vmatrix} -3 & -1 \\ 5 & 1 \end{vmatrix}, & \begin{vmatrix} -3 & -1 \\ 5 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}, & \begin{vmatrix} 2 & 2 \\ 5 & 1 \end{vmatrix}, & -\begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix}, & -\begin{vmatrix} 2 & 2 \\ -3 & -1 \end{vmatrix}, & \begin{vmatrix} 2 & 1 \\ -3 & -1 \end{vmatrix} \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2 & -1 \\ 3 & -8 & 1 \\ 1 & -4 & 1 \end{pmatrix} \end{aligned}$$

   The adjoint of A is simply the transpose of this matrix, i.e.

$$adj(A) = cof(A)^T = \begin{pmatrix} 1 & 3 & 1 \\ -2 & -8 & -4 \\ -1 & 1 & 1 \end{pmatrix}$$

(c) Compute $A \cdot adj(A)$

Solution: If we work out the multiplication, we get

$$A \cdot adj(A) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

(d) Compare the result in part (c) to that of part (a), and interpret the results. Is what you got a coincidence, or is there a reason behind it?

Solution: Notice that the matrix we get in part (c) is equal to $-2I = \det(A) \cdot I$.

This is not a coincidence. We know that if A is invertible, then $A^{-1} = \frac{1}{\det(A)} adj(A)$. If we multiply both sides on the left by A, we get $I = \frac{1}{\det(A)} A \cdot adj(A)$. Multiplying both sides by $\det(A)$ we get $\det(A) \cdot I = A \cdot adj(A)$.

2. Let A, B, C be 3×3 matrices such that $\det(A) = \det(B)$ and $\det(C) = 2$.

   (a) Find $\det((2A^{-1}C^2B^T)^T)$.

   Solution: Recall some common facts about determinants (where by X, Y we denote $n \times n$ matrices and by a we denote a scalar number.)

   i. $\det(X^T) = \det(X)$

   ii. $\det(XY) = \det(X)\det(Y)$

   iii. $\det(aX) = a^n \det(X)$

   iv. $\det(X^{-1}) = \frac{1}{\det(X)}$

   We apply the above four facts:

$$\begin{aligned}
\det((2A^{-1}C^2B^T)^T) &= \det(2A^{-1}C^2B^T) && \text{by } (i) \\
&= \det(2A^{-1})\det(C^2)\det(B^T) && \text{by } (ii) \\
&= 2^3\det(A^{-1})\det(C)^2\det(B) && \text{by } (iii), (ii), (i) \\
&= 2^3\frac{1}{\det(A)}2^2\det(A) && \text{by } (iv)\text{, assumptions} \\
&= 2^5 = 32
\end{aligned}$$

   (b) Find $\det(C^{-1} + adj(C))$.

   Solution: We notice that $adj(C) = \det(C)C^{-1} = 2C^{-1}$.

   Therefore,

$$\det(C^{-1} + adj(C)) = \det(3C^{-1}) = 3^3\det(C^{-1}) = \frac{27}{\det(C)} = \frac{27}{2}$$

2 Diagonalization

1. If possible, find an invertible matrix P and diagonal D such that $A = PDP^{-1}$, where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & -3 \\ 1 & -1 & 0 \end{pmatrix}$$

Solution: First we find the characteristic polynomial of A:

$$\begin{aligned} char(x) &= \det(A - xI) \\ &= \begin{vmatrix} 1-x & 0 & 0 \\ 1 & 2-x & -3 \\ 1 & -1 & -x \end{vmatrix} \\ &= (1-x)\begin{vmatrix} 2-x & -3 \\ -1 & -x \end{vmatrix} \\ &= (1-x)((2-x)(-x)-3) \\ &= (1-x)(x^2-2x-3) \\ &= (1-x)(x+1)(x-3) \end{aligned}$$

Therefore, our eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 3$. Because these numbers are distinct, we immediately know that A **is** in fact diagonalizable.

Next, we must solve each system of equations $(A - \lambda I)X = 0$.

(a) For $\lambda_1 = 1$:

$$A - I = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & -3 \\ 1 & -1 & -1 \end{pmatrix}$$

Using Gaussian elimination, this reduces to the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{pmatrix}$$

which gives solution $x = 2z$, $y = z$ with parameter $t = z$, i.e.

$$X = t(2,1,1)^T$$

so one particular eigenvector is $X_1 = (2,1,1)^T$.

(b) For $\lambda_2 = -1$:

$$A + I = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & -3 \\ 1 & -1 & 1 \end{pmatrix}$$

Using Gaussian elimination, this reduces to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

which gives solution $x = 0$, $y = z$ with parameter $t = z$, i.e.

$$X = t(0, 1, 1)^T$$

so one particular eigenvector is $X_2 = (0, 1, 1)^T$.

(c) For $\lambda_3 = 3$: Solving the system $(A - 3I)X = 0$ in the same manner as above gives us a third eigenvector $X_3 = (0, -3, 1)^T$.

Therefore, putting all this information together, we can write

$$P = (X_1\ X_2\ X_3) = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}$$

and

$$D = diag(\lambda_1, \lambda_2, \lambda_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

We may verify that these matrices satisfy the desired conditions.

3 Proofs

1. Let A, B denote symmetric matrices.

   (a) If $AB = BA$, show that AB is symmetric.

   Proof: What we **know** is that (i) $A = A^T$, (ii) $B = B^T$, (iii) $AB = BA$. What we **want** to show is that $(AB)^T = AB$.

   Starting from the left hand side...

$$\begin{aligned} (AB)^T &= B^T A^T && \text{by property of transpose} \\ &= BA && \text{because } A, B \text{ are symmetric} \\ &= AB && \text{by assumption} \end{aligned}$$

   (b) If AB is symmetric, show that $AB = BA$.

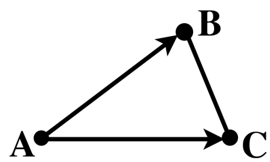
   Proof: What we **know** is that (i) $A = A^T$, (ii) $B = B^T$, (iii) $(AB)^T = AB$. What we **want** to show is that $AB = BA$.

   Starting from the right hand side...

$$\begin{aligned} BA &= B^T A^T && \text{because } A, B \text{ are symmetric} \\ &= (AB)^T && \text{by property of transpose} \\ &= AB && \text{by assumption} \end{aligned}$$

4 Vectors, lines and planes

1. Consider the three points $A(1,1,1)$, $B(1,2,3)$, $C(2,1,3)$. Find an equation of the plane containing these three points.



Solution: The normal vector of this plane is given by $\vec{n} = \vec{AB} \times \vec{AC}$.

First of all, $\vec{AB} = (1,2,3) - (1,1,1) = (0,1,2)$, and similarly $\vec{AC} = (1,0,2)$, so

$$\vec{n} = (0,1,2) \times (1,0,2) = \left(\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}, -\begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \right) = (2,2,-1)$$

In general, the equation of a plane is given by

$$\vec{n} \cdot (x,y,z) = ?$$

If we substitute $(x,y,z) = A = (1,1,1)$ into the above, we get

$$\vec{n} \cdot (1,1,1) = (2,2,-1) \cdot (1,1,1) = 3$$

so the equation of the plane we want is $(2,2,-1) \cdot (x,y,z) = 3$. Or by expanding,

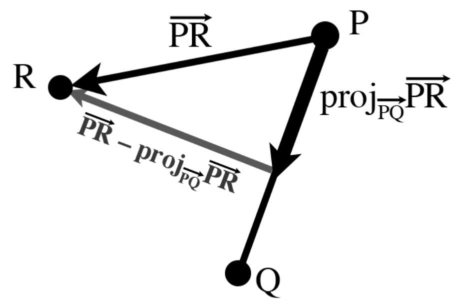
$$2x + 2y - z = 3$$

2. Find the minimal distance from the point $R(1,3,-1)$ to the line L defined by

$$(x,y,z) = (1,1,1) + t(2,-1,2)$$

Solution: By plugging in $t = 0$, we see that one point on L is $P = (1,1,1)$. By plugging in $t = 1$, another is $Q = (1,1,1) + (2,-1,2) = (3,0,3)$.

From the generic picture below, we can see that we want to find the length of the vector $\vec{PR} - proj_{\vec{PQ}}\vec{PR}$ as this will give us the answer.



Well, $\vec{PR} = (0,2,-2)$ and $\vec{PQ} = (2,-1,2)$.

So now we need to find $proj_{\vec{PQ}}\vec{PR}$:

$$proj_{\vec{PQ}}\vec{PR} = \frac{\vec{PR}\cdot\vec{PQ}}{\vec{PQ}\cdot\vec{PQ}}\vec{PQ} = \frac{-2-4}{4+1+4}(2,-1,2) = -\frac{2}{3}(2,-1,2)$$

So

$$\vec{PR} - proj_{\vec{PQ}}\vec{PR} = (0,2,-2) + \frac{2}{3}(2,-1,2) = \frac{1}{3}(4,4,-2).$$

Taking the length of this vector we get

$$\frac{1}{3}\sqrt{16+16+4} = 2$$

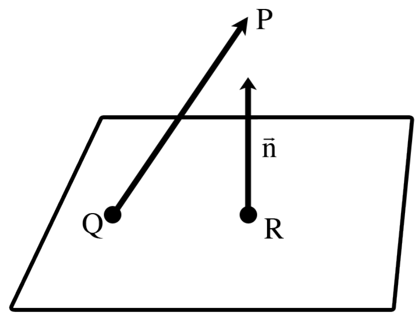
so the distance between the point R and the line L is 2.

3. Find the point R on the plane $2x - y + 3z = 1$ closest to the point $P = (1, 2, 3)$.

   Solution: From the equation of the plane, we can see that its normal vector is $\vec{n} = (2, -1, 3)$. We can also see that the point $Q = (x, y, z) = (0, -1, 0)$ satisfies the equation of the plane.

   If we draw the picture below (where R is the point we want to find, but don't know yet) we can observe that the vector $\vec{RP}$ is simply the projection $proj_{\vec{n}}\vec{QP}$, and we can evaluate this projection:

$$\vec{RP} = proj_{\vec{n}}\vec{QP} = \frac{\vec{QP} \cdot \vec{n}}{\vec{n} \cdot \vec{n}}\vec{n} = \frac{2 - 3 + 9}{4 + 1 + 9}\vec{n} = \frac{4}{7}\vec{n}$$



   Finally, we can find the coordinates of the point R by noting that

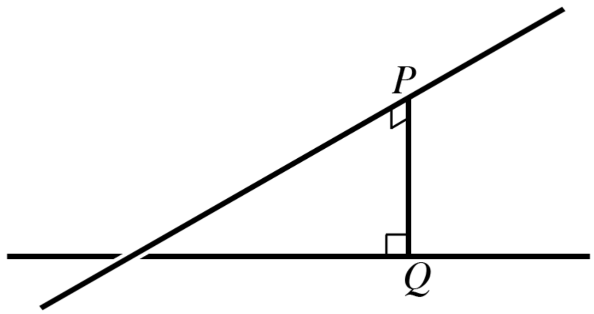
$$\vec{OR} = \vec{OP} + \vec{PR} = (1, 2, 3) - \frac{4}{7}(2, -1, 3) = \frac{1}{7}(-1, 18, 9)$$

4. Find the distance between the two lines

$$L_1 \; : \; (x,y,z) = (2,3,0) + t(1,-2,1)$$

$$L_2 \; : \; (x,y,z) = (3,2,1) + s(-1,1,1)$$

Solution: If we use P to denote points on L_1 and Q to denote points on L_2, we want to minimize the length of the vector $\vec{PQ}$. In order for the length to be minimized, $\vec{PQ}$ must be orthogonal to L_1 **and** at the same time orthogonal to L_2.



Mathematically, this means we must have

$$\vec{PQ} \cdot (1,-2,1) = 0 \tag{1}$$

$$\vec{PQ} \cdot (-1,1,1) = 0 \tag{2}$$

Because P lies on L_1, its coordinates are of the form

$$P = (2,3,0) + t(1,-2,1).$$

Similarly,

$$Q = (3,2,1) + s(-1,1,1)$$

so we have

$$\vec{PQ} = (1,-1,1) + s(-1,1,1) - t(1,-2,1) \tag{3}$$

Plugging this expression into equations (1) and (2) and simplifying, we get a system of linear equations

$$4 = 2s + 6t$$

$$1 = 3s + 2t$$

Solving this system yields a unique solution $s = -1/7$ and $t = 5/7$, so we may compute $\vec{PQ} = \frac{1}{7}(3,2,1)$, the length of which is $\frac{1}{7}\sqrt{9+4+1} = \sqrt{14}/7$.

5 Complex numbers

1. Find all complex numbers z such that $z^3 = 8i$.

   Solution: First we write $8i$ in polar coordinates:

   $$8i = 8e^{i\pi/2}$$

   Next we observe that we can make any number of full rotations by 2π degrees in the exponent, and the number above will be left unchanged, i.e. the following holds for any integer k:

   $$8e^{i\pi/2} = 8e^{i(\pi/2+2\pi k)}$$

   If we take the cube root of both sides of the equation $z^3 = 8i = 8e^{i(\pi/2+2\pi k)}$ we get

   $$z = 2e^{i(\pi/6+2\pi k/3)}$$

   Now we just need to plug in values of k:

   - For $k = 0$, we get $z = 2e^{i\pi/6}$,
   - For $k = 1$, we get $z = 2e^{i5\pi/6}$,
   - For $k = 2$, we get $z = 2e^{i3\pi/2}$,

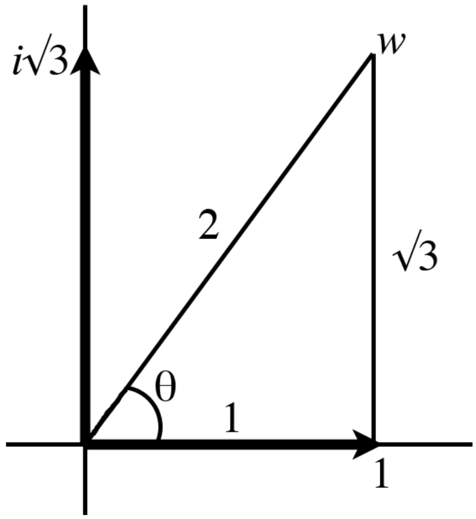
   which are the only three solutions.

2. Compute w^9 where $w = 1 + \sqrt{3}i$.

   Solution: First, we wish to write w in polar coordinates, e.g. $w = re^{i\theta}$, where r is the length of w and θ is the angle by which it is rotated in the plane.

   We can find r right away, by computing $r = ||w|| = \sqrt{1+3} = 2$.

   Now we only need to find θ. To do so, we first draw w in the plane, like so:



   By high school algebra, $\sin\theta = opp/hyp = \sqrt{3}/2$, which means that $\theta = \pi/3$.

   Therefore, we can write $w = 2e^{i\pi/3}$.

   In this form, we can easily compute $w^9 = 2^9 e^{3\pi i} = 512e^{\pi i} = 512(-1) = -512$.

6 Matrix transformations

1. Let T be a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^3$ such that

   - $T(1,1) = (5,-1,2)$
   - $T(0,-2) = (1,1,-4)$

   Determine $T(5,1)$.

   Solution: We must try to express the vector $X = (5,1)$ as a combination of the vectors $Y = (1,1)$ and $Z = (0,-2)$, in the form $X = aY + bZ$. We may do this either by solving a system of linear equations, or simply by observing that in order to get the "5" part of X, we must have $a = 5$ (because the first coordinate of Z is zero.)

   But $5Y = (5,5)$ so in order to get the "1" part of X we must subtract 4 from the second coordinate, i.e. we must set $b = 2$.

   Therefore, $X = 5Y + 2Z$.

   Now we may use the properties of a linear transformation to evaluate

$$T(X) = T(5Y + 2Z) = 5T(Y) + 2T(Z) = 5(5,-1,2) + 2(1,1,-4) = (27,-3,2).$$